Differential geometry of curves and surfaces

Week 1

Classical differential geometry: Study of curves and surfaces and their local (and global) properties using tools from Calculus.

Beginning: Fermat, Newton, Leibniz, Clairaut, Euler:
Extrinsic study of curves and surfaces
(17^{th}–18^{th} century)

Gauss–Riemann (19^{th} century): Intrinsic geometric properties

⟶ Development of general relativity, manifold theory, differential topology

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Tools that we will use in this class:

Linear algebra, Multivariable calculus, vector calculus
(but also some topology, group theory, ...)

Euclidean spaces

Def: Let V be a real vector space of dimension n.

1) A norm $\|\cdot\|$ on V: $\|\cdot\|: V \longrightarrow \mathbb{R}$ such that:

1) $\|x\| \geq 0$, "=" if and only if $x=0$

2) $\|\lambda \cdot x\| = |\lambda| \cdot \|x\|$

3) $\|x+y\| \leq \|x\| + \|y\|$ (triangle inequality)

- The set $\{x : \|x\| \leq 1\}$: is <u>convex</u>

2) (Scalar) inner product: $\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{R}$ such that it's:
   - <u>bilinear</u>
   - <u>symmetric</u>
   - positive definite: $\langle x, x \rangle \geq 0$ with "=" if and only if $x = 0$

- Standard inner product on $\mathbb{R}^n$:

$$\langle x, y \rangle = x_1 y_1 + \dots + x_n y_n = \sum_{i=1}^{n} x_i y_i$$

Norm associated to $\langle \cdot, \cdot \rangle$: $\|x\| = \sqrt{\langle x, x \rangle}$
("Euclidean norm")

- The set $\{x : \|x\| \leq 1\}$ in this case: ellipsoid.

For the rest of the course we will only consider norms from inner products

- $(V, \langle \cdot, \cdot \rangle)$: Euclidean space

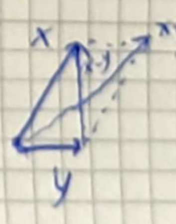
From now on: We will use the notation $\mathbb{E}^n$ for the n-dimensional Euclidean space.

<u>Basic properties</u>:

- Polarization identities:

$$\langle x, y \rangle = \frac{1}{2}\left(\|x+y\|^2 - \|x\|^2 - \|y\|^2\right)$$

$$= \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right)$$



- Parallelogramm law: $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + \|y\|^2$
(true only for Euclidean norms)

- One can always find an <u>orthonormal basis</u> $\{e_1, \dots, e_n\}$:

$$\langle e_i, e_j \rangle = \delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases} \qquad \text{(Kronecker symbol)}$$

- Cauchy - Schwarz inequality:

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|, \qquad \text{"="} \text{ if and only if } x \parallel y.$$

<u>Proof</u>:

Assume $x \neq 0$ (otherwise trivial).

Set $\tilde{y} = y - \lambda x$,

Choose λ so that $\tilde{y} \perp x$

$\Rightarrow \lambda = \dfrac{\langle x, y \rangle}{\langle x, x \rangle} = \dfrac{\langle x, y \rangle}{\|x\|^2}$

Then $0 \leq \|\tilde{y}\|^2$

$= \|y\|^2 - 2\lambda \langle x, y \rangle + \lambda^2 \|x\|^2$ ①

Then: ① $\Rightarrow$

$0 \leq \|y\|^2 - \dfrac{\langle x, y \rangle^2}{\|x\|^2}$

$\blacksquare$

<u>Exercise</u>: Prove that $\|x\| = \sqrt{\langle x, x \rangle}$ indeed satisfies the conditions of being a norm (you will need the Cauchy - Schwarz inequality)

<u>More definitions</u>

1) Euclidean distance: $x, y \in \mathbb{E}^n$, $d(x, y) := \|x - y\|$.

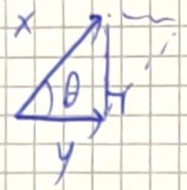
2) Angle between $x, y \neq 0$: $\theta \in [0, \pi]$,

$$\cos\theta := \frac{\langle x, y \rangle}{\|x\| \, \|y\|} \in [-1, 1] \text{ by Cauchy - Schwarz}$$

$$\rightsquigarrow \langle x, y \rangle = \|x\| \cdot \|y\| \cdot \cos\theta$$

3) Area of parallelogram spanned by x, y:

$$\text{Area}(P(x,y)) = \|x\| \cdot \|y\| \cdot \sin\theta = \sqrt{\|x\|^2\|y\|^2 - \langle x,y\rangle^2}$$

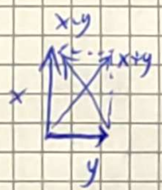


4) Orthogonal vectors: $x \perp y \Leftrightarrow \langle x,y\rangle = 0$.

Proposition:

The following condition are equivalent:

1) $x \perp y$
2) $\|x \pm y\|^2 = \|x\|^2 + \|y\|^2$ (Pythagoras theorem)
3) $\|x+y\| = \|x-y\|$



Lemma: $\mathbb{E}^n$ equiped with $d(\cdot,\cdot)$ is a metric space,

i.e:
- $d(x,y) \geq 0$, "=" iff $x=y$
- $d(x,y) = d(y,x)$
- $d(x,z) \leq d(x,y) + d(y,z)$.

Orientation of vector spaces

Let $\{u_1, \dots, u_n\}$ and $\{w_1, \dots, w_n\}$ be two bases for the vector space V. We can express each vector u_i in the w_j basis:

$$u_i = \sum_{j=1}^{n} p_{ij} w_j.$$ The matrix $P = [p_{ij}]$: invertible

(Matrix of change of basis

If $x = \sum_{i=1}^{n} x_i u_i = \sum_{j=1}^{n} x_j' w_j$, so $x_j' = \sum_{i=1}^{n} p_{ij} x_i$

coordinates in $\{u\}$

coordinates in $\{w\}$

then, as column vectors,

$$\begin{pmatrix} x_1' \\ \vdots \\ x_n' \end{pmatrix} = P^T \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix})$$

Recall:
If $A = \begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \\ \vdots & & \ddots \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix}$

then
$$(A \cdot x)_i = \sum_{j=1}^{n} A_{ij} x_j$$

<u>Definition</u>: The bases $\{u_1, \dots, u_n\}$ and $\{w_1, \dots, w_n\}$ have the <u>same orientation</u> if $\det P > 0$.

(opposite orientation otherwise)

So: Two classes of bases, in each class: bases have the same orientation

<u>Def</u>: An orientation on V: A choice of one of this class of bases.

We usually call the chosen orientation "positive", the opposite one "negative"

Eg: On $\mathbb{R}^2$: $\{(1,0), (0,1)\}$ and $\{(1,0), (0,-1)\}$ have <u>opposite orientation</u>.

<u>Def</u>: A linear invertible map $A: V \to V$ <u>preserves</u> orientation if $\det A > 0$.

Define: $GL_+(V) = \{A: V \to V$ linear & invertible, $\det A > 0\}$

Connected subgroup of $GL(V) = \{A: V \to V$: linear & invertible$\}$

Similarity transformations and isometries

Def:
- Similarity map of scale $\lambda > 0$:

$$f: \mathbb{E}^n \longrightarrow \mathbb{E}^n \quad \text{s.t.} \quad d(f(x), f(y)) = \lambda d(x,y)$$

- Isometry: Similarity map with $\lambda = 1$
  (so it preserves distances)

Properties:
- A similarity map must be continuous and bijective.
- Similarity maps form a group under composition
  - Isometries: Normal subgroup.

Lemma: If $g: \mathbb{E}^n \to \mathbb{E}^n$ is an isometry with $g(0) = 0$ then:
- $\langle g(x), g(y) \rangle = \langle x, y \rangle \quad \forall x, y \in \mathbb{E}^n$
- g is linear
- With respect to any orthonormal basis:
  $g(x) = Ax$ where the matrix A satisfies $A^T A = I$

Proof: Since $\forall x,y \quad \|g(y) - g(x)\| = \|y - x\| \xRightarrow[g(0)=0]{x=0} \|g(y)\| = \|y\| \quad \forall y$

Using one of the polarization identities:

$$\langle g(x), g(y) \rangle = \frac{1}{2}\left(\|g(x)\|^2 + \|g(y)\|^2 - \underbrace{\|g(x) - g(y)\|^2}_{d(x,y)^2}\right) =$$

$$= \frac{1}{2}\left(\|x\|^2 + \|y\|^2 - \|x - y\|^2\right)$$

$$= \langle x, y \rangle$$

To prove g is a linear map. We need to show that $g(\lambda x) = \lambda g(x)$ $\forall \lambda \in \mathbb{R}$, $x \in E^n$

and $g(x+y) = g(x) + g(y)$.

- $\|g(\lambda x) - \lambda g(x)\|^2 = \|g(\lambda x)\|^2 - 2\langle g(\lambda x), \lambda g(x)\rangle + \|\lambda g(x)\|^2$
$= \|g(\lambda x)\|^2 - 2\lambda \langle g(\lambda x), g(x)\rangle + \lambda^2 \|g(x)\|^2$
$= \|\lambda x\|^2 - 2\lambda \langle \lambda x, x\rangle + \lambda^2 \|x\|^2 = 0$

- $\|g(x+y) - g(x) - g(y)\|^2 = \|g(x+y)\|^2 + \|g(x)\|^2 + \|g(y)\|^2 - 2\langle g(x+y), g(x)\rangle$
$- 2\langle g(x+y), g(y)\rangle + 2\langle g(x), g(y)\rangle$
$= \|x+y\|^2 + \|x\|^2 + \|y\|^2 - 2\langle x+y, x\rangle$
$- 2\langle x+y, y\rangle - 2\langle x, y\rangle = 0$

So g is linear.

If $\{e_1, \dots, e_n\}$ is an orthonormal basis, and A is the matrix representation of g in this basis:

- $\{e_1, \dots, e_n\}$ orthonormal: $\langle e_i, e_j\rangle = \delta_{ij}$
- g isometry: $\langle Ae_i, Ae_j\rangle = \langle e_i, e_j\rangle = \delta_{ij}$

Recall: $(A \cdot x)_i = \sum_{j=1}^{n} A_{ij} x_j$ and, if A, B are matrices: $(A \cdot B)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$

So $(A e_i)_k = A_{ik}$ (since $(e_i)_j = \begin{cases} 1, & i=j \\ 0, & \text{else} \end{cases}$)

and $\langle Ae_i, Ae_j\rangle = \sum_{k=1}^{n} (Ae_i)_k \cdot (Ae_j)_k = \sum_{k=1}^{n} A_{ik} \cdot A_{jk} = \sum_{k=1}^{n} A_{ik} \cdot (A^T)_{kj}$

Since $\langle Ae_i, Ae_j\rangle = \delta_{ij}$: This means that

$A \cdot A^T = \mathbb{I}$ so A^T is the inverse of A (hence, also $A^T A = \mathbb{I}$)

$\square$

Corollary:

A similarity transformation $T: \mathbb{E}^n \to \mathbb{E}^n$ of scale $\lambda > 0$ is of the form $T(x) = \lambda g(x) + b$, where

- $g: \mathbb{E}^n \to \mathbb{E}^n$ is an isometry with $g(0) = 0$.
- $b = T(0) \in \mathbb{E}^n$

In particular, T is an <u>affine map</u>.

<u>Remark</u>: In what follows, we will denote with

$$GL(V), \quad GL_+(V), \quad SL(V) = \{A \in GL(V) : \det A = +1\}$$

the groups of <u>linear maps</u> of V on itself, while

$$GL_n(\mathbb{R}), \quad GL_{n+}(\mathbb{R}), \quad SL_n(\mathbb{R})$$

will denote the corresponding <u>groups of $n \times n$ matrices</u>

The two are, of course, in 1-1 correspondence via the choise of a basis.

Orthogonal group:

Orthogonal group of matrices:

$$O(n) = \{ A \in M_n(\mathbb{R}) : A^T A = I_n \}$$

If $A \in O(n)$: $\det A = \pm 1$

$$SO(n) = \{ A \in O(n) : \det A = +1 \} \quad \text{(preserve orientation)}$$

Lemma: The following conditions are equivalent:

1) $A \in O(n)$,
2) $A \in GL_n$ and $A^{-1} = A^T$
3) $\|Ax\| = \|x\| \quad \forall x \in \mathbb{R}^n$
4) $\langle Ax, Ay \rangle = \langle x, y \rangle \quad \forall x, y \in \mathbb{R}^n$
5) The rows of A are an orthonormal basis
6) The columns " " " " "

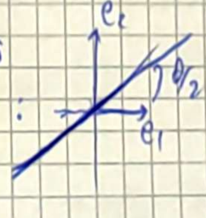
- Orthogonal group in 2d:

Proposition: If $A \in O(2)$, then $\exists\, \theta \in [0, 2\pi)$ such that

- $A = \underbrace{R_\theta}_{\text{Rotation by angle } \theta} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, if $\det A = +1$

- $A = \underbrace{S_{\theta/2}}_{\text{Reflection across the axis}} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$, if $\det A = -1$

Reflection across the axis:



Proof: The columns of A are an orthonormal basis of $\mathbb{E}^2$ (since $A \in O(2)$)

For the first column: It's a unit vector, so $\exists\, \theta \in [0, 2\pi)$ s.t. it's of the form $\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$

The second column: Unit size and orthogonal to the first, so it has to be $\pm \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$. $\boxed{//}$

- Orthogonal group in 3d:

Theorem (Euler): If $A \in SO(3)$, then either $A = I$ or there exists an axis and an angle $\theta \in [0, 2\pi)$ such that A is the rotation by θ around the axis.

Proof:

It suffices to show that A has an eigenvector v with eigenvalue $+1$. Then, A will restrict to an isometry on the orthogonal complement $v^\perp$ which is 2-dimensional. By the previous discussion, it should be a rotation there.

Note: $A^T A = I$ and $\det A = +1$

So we compute: $\det(A - I) = \det(A - A^T A) = \det((I - A^T)A)$

$= \det(I - A^T) \cdot \det A = \det(I - A)$

$\overset{3\text{ dimensions}}{=} -\det(A - I)$

So $\det(A-\mathbb{I}) = -\det(A-\mathbb{I}) \Leftrightarrow \det(A-\mathbb{I}) = 0$

So $A-\mathbb{I}$ has a kernel. Let $\overset{\neq 0}{v} \in \mathrm{Ker}(A-\mathbb{I}) \Rightarrow Av = v$.

The orthogonal complement $v^\perp = \{x \in \mathbb{E}^3 : x \perp v\}$ is a 2-d subspace of $\mathbb{E}^3$ (going through 0) that is preserved by A, since if $x \in v^\perp$: $0 = \langle x, v\rangle \overset{A\in O(3)}{=} \langle Ax, Av\rangle \overset{Av=v}{=} \langle Ax, v\rangle$ so we have $Ax \in v^\perp$ as well.

So $A: v^\perp \longrightarrow v^\perp$ is a linear map, which satisfies $\langle Ax, Ay\rangle = \langle x, y\rangle$

So it's a 2-d linear isometry. By the previous discussion, if $\{u_1, u_2\}$ is an orthonormal basis of $v^\perp$ (hence, $\{u_1, u_2, v\}$ is an orthonormal basis of $\mathbb{E}^3$), we have that, when transformed in the basis $\{u_1, u_2\}$, $A|_{v^\perp}$ becomes

$$\tilde{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \text{ or } \tilde{A} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \text{ for some } \theta.$$

So in the basis $\{u_1, u_2, v\}$ of $\mathbb{E}^3$ we have that A becomes (since $Av = v$)

$$\tilde{A} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ or } \tilde{A} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

From the above, only the first case has determinant $+1$.
The above is the expression of a rotation around v by angle θ. When $\theta \neq 0$, the axis is unique. $\blacksquare$

Prop: $\mathrm{Tr}(A) = 1 + 2\cos\theta$

(So we can compute the angle of rotation without transforming the basis)

Remark: From the previous discussion:

- $SO(2)$ is "1 dimensional" (parametrized by 1 continuous parameter, the angle of rotation).

  In fact: $SO(2) \simeq \mathbb{S}^1$ (unit circle)

- $SO(3)$ is "3-dimensional" (parametrized by axis direction and magnitude of rotation)

  In fact $SO(3) \simeq$ 3 dimensional projective space.

Euler angles: If x, z are two orthogonal axes in $\mathbb{E}^3$, any $A \in SO(3)$ can be written as the product of three rotations

$$A = \underbrace{R^{(z)}_{\theta_1}}_{\text{rotation around } z \text{ by } \theta_1} \cdot R^{(x)}_{\theta_2} \cdot R^{(z)}_{\theta_3}$$

Vector calculus (or vectorial geometry)

Let $\mathbb{E}^3$ be an oriented Euclidean space

Def: For any $x, y \in \mathbb{E}^3$, we define $x \times y \in \mathbb{E}^3$ by the following conditions:

i) $x \times y \perp x$ and $x \times y \perp y$

ii) $\| x \times y \| = \text{Area}(P(x,y))$

iii) If x, y are linearly independent: $\{x, y, x \times y\}$ is a positively oriented basis

The above: Defines a unique vector
(unique direction by (i) and (iii),
magnitude by (ii))

But in order to perform calculations, we need a coordinate expression:

<u>Proposition</u>
If $\{e_1, e_2, e_3\}$ is a positively oriented orthonormal basis,

and $x = x_1 e_1 + x_2 e_2 + x_3 e_3$

$y = y_1 e_1 + y_2 e_2 + y_3 e_3$

Then $x \times y = (x_2 y_3 - x_3 y_2)\, e_1 + (x_3 y_1 - x_1 y_3)\, e_2 + (x_1 y_2 - x_2 y_1)\, e_3$

$$= \det\begin{pmatrix} x_2 & y_2 \\ x_3 & y_3 \end{pmatrix} e_1 - \det\begin{pmatrix} x_1 & y_1 \\ x_3 & y_3 \end{pmatrix} e_2 + \det\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} e_3$$

<u>Proof</u>: One has to check that the above expression verifies conditions (i), (ii), (iii) (see Exercises) ▨

Note, schematically:

$$x \times y = \det\begin{pmatrix} x_1 & y_1 & e_1 \\ x_2 & y_2 & e_2 \\ x_3 & y_3 & e_3 \end{pmatrix}$$

(The cross product operator: bilinear, <u>antisymmetric</u>)

<u>Def</u>: Mixed product: If $x, y, w \in E^3$,

$$[x, y, w] := \langle x \times y, w \rangle \underset{\text{in orthonormal basis}}{=} \det\begin{pmatrix} x_1 & y_1 & w_1 \\ x_2 & y_2 & w_2 \\ x_3 & y_3 & w_3 \end{pmatrix}$$

Remark:

$[x, y, w]$ is the oriented (or signed) volume of the parallelepipede $P(x, y, w)$